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Popov–Belevitch–Hautus type controllability tests for linear complementarity systems[☆]

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Abstract

It is well-known that checking certain controllability properties of very simple piecewise linear systems are undecidable problems. This paper deals with the controllability problem of a class of piecewise linear systems, known as linear complementarity systems. By exploiting the underlying structure and employing the results on the controllability of the so-called conewise linear systems, we present a set of inequality-type conditions as necessary and sufficient conditions for controllability of linear complementarity systems. The presented conditions are of Popov–Belevitch–Hautus type in nature.

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1. Introduction

Consider a finite-dimensional continuous-time linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and A, B are matrices with suitable sizes. Controllability property of such a system refers to the fact that any initial state can be steered to any final state by choosing the input appropriately. This notion was introduced by Kalman [14] and was extensively studied by Kalman himself [15] and many others (see [13,24] for historical details) in the early sixties. The well-known Kalman's rank condition

$$\text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n \quad (2)$$

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is necessary and sufficient for the controllability of (1). An alternative characterization, which is sometimes called Popov–Belevitch–Hautus (PBH) test, presents an equivalent condition in terms of the *eigenmodes* of the system:

$$\lambda \in \mathbb{C}, \ z \in \mathbb{C}^n, \ z^*A = \lambda z^*, \ z^*B = 0 \Rightarrow z = 0. \quad (3)$$

An interesting and useful variant is the constrained controllability, i.e. controllability with a restricted set of inputs such as bounded or nonnegative inputs. Early work in this direction considers only constraint sets which contain the origin in their interior [17, Theorem 8, p. 92]. However, the constraint set does not contain the origin in its interior in many interesting cases, for instance, when only nonnegative controls are allowed. Saperstone and Yorke [19] were the first to consider constraint sets that do not have the origin in their interior. In particular, they considered the case for which the inputs are constrained to the set $[0, 1]$. More general constraint sets were studied by Brammer [4]. In particular, he showed that a linear system (1) with the constraint $u(t) \geq 0$ for all t is controllable if, and only if, the implications (3) and

$$\lambda \in \mathbb{R}, \ z \in \mathbb{R}^n, \ z^T A = \lambda z^T, \ z^T B \geq 0 \Rightarrow z = 0. \quad (4)$$

are satisfied.

When one leaves the realm of linear systems, characterization of controllability becomes a hard problem. Typical controllability related results for nonlinear systems are local, i.e. only valid in a neighborhood of the initial state. Global controllability problem is known to be an NP-hard problem for classes of bilinear systems [23]. Also for piecewise linear systems, certain controllability problems are known to be quite complex problems. Consider, for instance, discrete-time sign-systems of the form

$$x_{t+1} = \begin{cases} A_-x_t + bu_t & \text{if } c^T x_t < 0, \\ A_0x_t + bu_t & \text{if } c^T x_t = 0, \\ A_+x_t + bu_t & \text{if } c^T x_t > 0. \end{cases} \quad (5)$$

Blondel and Tsitsiklis [3] showed that checking the null-controllability property (meaning that any initial state can be steered to the zero state) of these systems is an *undecidable* problem.

Despite this pessimistic result, controllability problems for hybrid systems have received considerable attention. Lee and Arapostathis [16] looked at the controllability of a class of “hypersurface systems”. They provide conditions that are not stated in an easily verifiable form. Bemporad et al. [2] take an algorithmic approach based on optimization tools. Their approach makes it possible to check controllability of a given (discrete-time) system. However, it does not allow drawing conclusions about any class of systems as in the current paper. In a recent paper, Brogliato gives necessary and sufficient conditions for global controllability of a class of piecewise linear systems [5]. This work is based on a case-by-case analysis and applies only to the planar case. Nesic [18] and Smirnov [22, Chapter 6] obtain characterizations of controllability that apply to some classes of piecewise linear systems. All these works [5,18,22] consider systems that are different from those we look at in this paper.

Starting with [6], we have looked at the controllability properties of piecewise linear systems with some additional structure. In [6], necessary and sufficient conditions were presented for planar bimodal piecewise linear systems with a continuous vector field. Later, we generalized these results to bimodal systems with arbitrary state space dimension in [7]. In [8], these results were further generalized to conewise linear systems (CLSs), i.e. piecewise linear systems for which the state space is partitioned into solid polyhedral cones and on each of these cones a linear dynamics is active. A common feature of this line of research is the use of combination of ideas from geometric control theory and mathematical programming. The nature of the established conditions resembles very much the PBH test. Moreover, both Kalman’s and Brammer’s results can be recovered as particular cases.

The aim of this paper is to address the controllability problem for yet another class of piecewise linear systems, namely linear complementarity systems. By using the ideas and results of [8], we will present compact necessary and sufficient conditions that are of PBH type. The structure of the paper is as follows. This section ends with notational conventions. In the next section, we introduce the linear complementarity problem/

system and discuss some special cases. The main results of the paper are presented and proved in Section 3. The paper closes with conclusions in Section 4. A very brief review of basic geometric control theory is included in Appendix A for the sake of completeness.

1.1. Notation

The symbol \mathbb{R} denotes the set of real numbers, \mathbb{R}^n n -tuples of real numbers, $\mathbb{R}^{n \times m}$ $n \times m$ real matrices, \mathbb{C} the set of complex numbers, and \mathbb{C}^n n -tuples of complex numbers. For a matrix $A \in \mathbb{R}^{n \times m}$, A^T stands for its transpose, A^{-1} for its inverse (if exists), $\text{im } A$ for its image, i.e. the set $\{y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m\}$. We write A_{ij} for the (i, j) th element of A . For $\alpha \subseteq \{1, 2, \dots, n\}$, and $\beta \subseteq \{1, 2, \dots, m\}$, $A_{\alpha\beta}$ denotes the submatrix $\{A_{jk}\}_{j \in \alpha, k \in \beta}$. If $\alpha = \{1, 2, \dots, n\}$ ($\beta = \{1, 2, \dots, m\}$), we also write $A_{\bullet\beta}$ ($A_{\alpha\bullet}$). Inequalities for vectors must be understood componentwise. Similarly, max operator acts on the vectors componentwise. We write $x \perp y$ if $x^T y = 0$. For a subspace \mathcal{X} of \mathbb{R}^n , \mathcal{X}^\perp denotes the orthogonal subspace of \mathcal{X} , i.e. the subspace $\{y \mid y^T x = 0 \text{ for all } x \in \mathcal{X}\}$. The asterisk symbol will have three different meanings: For a complex vector $z \in \mathbb{C}^n$, z^* denotes its conjugate transpose. For a nonempty set \mathcal{Q} , \mathcal{Q}^* stands for its dual cone, i.e. the set $\{x \mid x^T y \geq 0 \text{ for all } y \in \mathcal{Q}\}$. Also, it will be used to indicate minimal/maximal elements of classes of subspace (see Appendix A).

2. Linear complementarity problem/system

The problem of finding a vector $z \in \mathbb{R}^m$ such that

$$z \geq 0, \quad (6a)$$

$$q + Mz \geq 0, \quad (6b)$$

$$z^T(q + Mz) = 0 \quad (6c)$$

for a given vector $q \in \mathbb{R}^m$ and a matrix $M \in \mathbb{R}^{m \times m}$ is known as the linear complementarity problem. We denote (6) by $\text{LCP}(q, M)$. It is well-known [10, Theorem 3.3.7] that the $\text{LCP}(q, M)$ admits a unique solution for each q if, and only if, M is a P -matrix, i.e. all its principal minors are positive. It is also known that z depends on q in a Lipschitz continuous way in this case.

Linear complementarity systems consist of nonsmooth dynamical systems that are obtained in the following way. Take a standard linear input/output system. Select a number of input/output pairs (z_i, w_i) , and impose for each of these pairs complementarity relation of the type (6) at each time t , i.e. both $z_i(t)$ and $w_i(t)$ must be nonnegative, and at least one of them should be zero for each time instant $t \geq 0$. This results in a dynamical system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ez(t), \quad (7a)$$

$$w(t) = Cx(t) + Du(t) + Fz(t), \quad (7b)$$

$$0 \leq z(t) \perp w(t) \geq 0, \quad (7c)$$

where $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ is the state, $z, w \in \mathbb{R}^k$ are the complementarity variables, and all the matrices are of appropriate sizes. A wealth of examples and application areas of LCSs can be found in [9,12,20,21].

A set of standing assumptions throughout this paper are the following.

Assumption 2.1. The following conditions are satisfied for the LCS (7)

- (1) The matrix F is a P -matrix.
- (2) The transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix.

Admittedly, these are restrictive assumptions within the general class of LCSs. The first one rules out many interesting instances of LCSs whereas the second one requires that the number of inputs and the number of complementarity variables be the same, i.e. $k = m$.

It follows from Assumption 2.1(1) that $z(t)$ is a piecewise linear function of $Cx(t) + Du(t)$ (see e.g. [10]). This means that for each initial state x_0 and locally integrable input u there exist a unique absolutely continuous state trajectory $x^{x_0,u}$ and locally integrable trajectories $(z^{x_0,u}, w^{x_0,u})$ such that $x^{x_0,u}(0) = x_0$ and the triple $(x^{x_0,u}, z^{x_0,u}, w^{x_0,u})$ satisfies the relations (7) for almost all $t \geq 0$.

We say that the LCS (7) is (*completely*) *controllable* if for any pair of states $(x_0, x_f) \in \mathbb{R}^{n+n}$ there exists a locally integrable input u such that the trajectory $x^{x_0,u}$ satisfies $x^{x_0,u}(T) = x_f$ for some $T > 0$.

In two particular cases, one can employ the available results for the linear systems to determine whether (7) is controllable.

2.1. Linear systems

Consider the LCS

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (8a)$$

$$w(t) = u(t) + z(t), \quad (8b)$$

$$0 \leq z(t) \perp w(t) \geq 0. \quad (8c)$$

It can be verified that Assumption 2.1 holds. Note that this system is controllable if, and only if, the linear system (8a) is controllable. In turn, this is equivalent to the implication

$$\lambda \in \mathbb{C}, \quad z \in \mathbb{C}^n, \quad z^*A = \lambda z^*, \quad z^*B = 0 \Rightarrow z = 0. \quad (9)$$

In this case, we say that the pair (A, B) is controllable.

2.2. Linear systems with nonnegative inputs

Consider the LCS

$$\dot{x}(t) = Ax(t) + Bu(t) + Bz(t), \quad (10a)$$

$$w(t) = u(t) + z(t), \quad (10b)$$

$$0 \leq z(t) \perp w(t) \geq 0. \quad (10c)$$

Note that the solution to the LCP (10b)–(10c) can be given as $z(t) = u^-(t)$ and $w(t) = u^+(t)$ where $\xi^+ := \max(\xi, 0)$ and $\xi^- := \max(-\xi, 0)$ denote the nonnegative and nonpositive part of the real vector $\xi = \xi^+ - \xi^-$, respectively.

Therefore, this LCS is controllable if, and only if, the linear system

$$\dot{x}(t) = Ax(t) + Bv(t)$$

with the input constraint $v(t) \geq 0$ is controllable. It follows from [4, Corollary 3.3] that this system is controllable if, and only if, the following two conditions hold:

- (1) the pair (A, B) is controllable,
- (2) the implication

$$\begin{aligned} \lambda \in \mathbb{R}, \quad z \in \mathbb{R}^n, \quad z^T A = \lambda z^T, \\ z^T B \geq 0 \Rightarrow z = 0 \end{aligned} \quad (11)$$

holds.

3. Main results

The following theorem presents algebraic necessary and sufficient conditions for the controllability of an LCS.

Theorem 3.1. Consider an LCS (7) satisfying Assumption 2.1. It is controllable if, and only if, the following two conditions hold

- (1) The pair $(A, [B \ E])$ is controllable.
- (2) The system of inequalities

$$\eta \geq 0, \quad (12a)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0, \quad (12b)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} E \\ F \end{bmatrix} \leq 0 \quad (12c)$$

admits no solution $\lambda \in \mathbb{R}$ and $0 \neq (\xi, \eta) \in \mathbb{R}^{n+m}$.

To prove this theorem, we review the controllability properties of a closely related system class: CLSs. A CLS is a dynamical system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(Cx(t) + Du(t)), \quad (13)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, all matrices are of appropriate sizes, and the function f is a *conewise linear function*, i.e. there exist an integer r , solid polyhedral cones \mathcal{Y}_i , and matrices $M^i \in \mathbb{R}^{n \times p}$ for $i = 1, 2, \dots, r$ such that $\bigcup_{i=1}^r \mathcal{Y}_i = \mathbb{R}^p$ and $f(y) = M^i y$ if $y \in \mathcal{Y}_i$.

Note that f is necessarily Lipschitz continuous. This means that for each initial state x_0 and locally integrable input u there exist a unique absolutely continuous state trajectory $x^{x_0,u}$ satisfying (13) with $x^{x_0,u}(0) = x_0$.

We say that the CLS (13) is (*completely*) *controllable* if for any pair of states $(x_0, x_f) \in \mathbb{R}^{n+n}$ there exists a locally integrable input u such that the trajectory $x^{x_0, u}$ satisfies $x^{x_0, u}(T) = x_f$ for some $T > 0$.

The controllability problem for these systems was treated in [8] where the following theorem was proven.

Theorem 3.2. *Consider the CLS (13) such that $p = m$ and the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. It is completely controllable if, and only if,*

(1) the relation

$$\sum_{i=1}^r \langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \mathbb{R}^n \quad (14)$$

is satisfied and

(2) the implication $\lambda \in \mathbb{R}, z \in \mathbb{R}^n, w_i \in \mathbb{R}^m$

$$[z^T \ w_i^T] \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0,$$

$$w_i \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \Rightarrow z = 0$$

holds.

Next, we show that a linear complementarity system satisfying Assumption 2.1(1) can be viewed as a CLS. To see this, note that it follows from (7b)–(7c) and Assumption 2.1(1) that there exists an index set $\alpha \subseteq \{1, 2, \dots, m\}$ such that

$$z_\alpha(t) = -F_{\alpha\alpha}^{-1}(C_{\alpha\bullet}x(t) + D_{\alpha\bullet}u(t)) \geq 0, \quad (15a)$$

$$z_{\alpha^c}(t) = 0, \quad (15b)$$

$$w_\alpha(t) = 0, \quad (15c)$$

$$w_{\alpha^c}(t) = (C_{\alpha^c\bullet} - F_{\alpha^c\alpha}F_{\alpha\alpha}^{-1}C_{\alpha\bullet})x(t) + (D_{\alpha^c\bullet} - F_{\alpha^c\alpha}F_{\alpha\alpha}^{-1}D_{\alpha\bullet})u(t) \geq 0, \quad (15d)$$

where $\alpha^c = \{1, 2, \dots, m\} \setminus \alpha$. Let Π^α be the permutation matrix with $\Pi^\alpha y = \text{col}(y_\alpha, y_{\alpha^c})$. Define

$$T^\alpha := \begin{bmatrix} -F_{\alpha\alpha}^{-1} & 0 \\ -F_{\alpha^c\alpha}F_{\alpha\alpha}^{-1} & I \end{bmatrix}$$

and $\mathcal{Y}_\alpha := \{y \in \mathbb{R}^m \mid T^\alpha \Pi^\alpha y \geq 0\}$. Note that $\text{col}(z_\alpha(t), w_{\alpha^c}(t)) = T^\alpha \Pi^\alpha (Cx(t) + Du(t))$. It follows from Assumption 2.1(1) that $\bigcup_\alpha \mathcal{Y}_\alpha = \mathbb{R}^m$. By defining $f(y) = -E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0] \Pi^\alpha y$ for $y \in \mathcal{Y}_\alpha$, we can rewrite (7) in the form (13).

A direct application of Theorem 3.2 results in the following corollary.

Corollary 3.3. *Consider the LCS (7). Suppose that Assumption 2.1 is satisfied. Define $M^\alpha = -E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0] \Pi^\alpha$ for each $\alpha \subseteq \{1, 2, \dots, m\}$. It is completely controllable if, and only if,*

(1) the relation

$$\sum_{\alpha \subseteq \{1, 2, \dots, m\}} \langle A + M^\alpha C \mid \text{im}(B + M^\alpha D) \rangle = \mathbb{R}^n \quad (16)$$

is satisfied and

(2) the implication

$$\lambda \in \mathbb{R}, \ z \in \mathbb{R}^n, \ w^\alpha \in \mathbb{R}^m,$$

$$[z^T \ (w^\alpha)^T] \begin{bmatrix} A + M^\alpha C - \lambda I & B + M^\alpha D \\ C & D \end{bmatrix} = 0,$$

$$w^\alpha \in \mathcal{Y}_\alpha^* \text{ for all } \alpha \subseteq \{1, 2, \dots, m\} \Rightarrow z = 0$$

holds.

Our goal is to show the equivalence of the conditions of this corollary and those of Theorem 3.1. To do so, we need the following auxiliary lemmas. The first one is about controllability properties of a number of linear systems that are obtained from a given linear system by applying certain output injections.

In the sequel, we will often employ some notions of geometric control theory. For the sake of completeness, a review of basic concepts and notations will be given in Appendix A.

Lemma 3.4. *Suppose that $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. Let A be a finite set and let $\{N^\alpha\}$ for $\alpha \in A$ be given matrices. Denote the subspaces $\mathcal{V}^*(A, B, C, D)$ and $\mathcal{T}^*(A, B, C, D)$ by \mathcal{V}^* and \mathcal{T}^* , respectively. Let $\pi_{\mathcal{V}^*}$ be the projection¹ on \mathcal{V}^* along \mathcal{T}^* and $\pi_{\mathcal{T}^*}$ be the projection on \mathcal{T}^* along \mathcal{V}^* . Then, the following statements are equivalent:*

(1) The implication

$$z^*(A + N^\alpha C) = \lambda z^*, \quad z^*(B + N^\alpha D) = 0$$

$$\text{for all } \alpha \in A \text{ and for some } \lambda \in \mathbb{C} \Rightarrow z = 0$$

holds.

(2) The equality

$$\langle \pi_{\mathcal{V}^*}(A - BK)\pi_{\mathcal{V}^*} \mid \sum_{\alpha \in A} [\text{im}(\pi_{\mathcal{V}^*}(A + N^\alpha C)\pi_{\mathcal{T}^*}) + \text{im}(\pi_{\mathcal{V}^*}(B + N^\alpha D))] \rangle = \mathcal{V}^* \quad (17)$$

holds for all $K \in \mathcal{K}(\mathcal{V}^*)$.

(3) The equality

$$\sum_{\alpha \in A} \langle A + N^\alpha C \mid \text{im}(B + N^\alpha D) \rangle = \mathbb{R}^n$$

holds.

Proof. $1 \Rightarrow 2$: Denote the subspace on the left-hand side of (17) by \mathcal{R} , $\pi_{\mathcal{V}^*}(A - BK)\pi_{\mathcal{V}^*}$ by \bar{A} , $\pi_{\mathcal{V}^*}(A + N^\alpha C)\pi_{\mathcal{T}^*}$ by \bar{B}_1 , and $\pi_{\mathcal{V}^*}(B + N^\alpha D)$ by \bar{B}_2 . Obviously, $\mathcal{R} \subseteq \mathcal{V}^*$. To show that the reverse inclusion holds, suppose that $\zeta \in \mathcal{R}^\perp$, i.e.

$$\zeta^T \bar{A}^k \bar{B}_i = 0 \quad (18)$$

for all $k \geq 0$ and $i = 1, 2$. We use the very same idea of Hautus' proof for the linear case (see [11]). Let ψ be a polynomial

¹ A mapping $\pi : \mathcal{X} \rightarrow \mathcal{X}$ is called a projection on \mathcal{X}_1 along \mathcal{X}_2 if $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\pi x = x$ for all $x \in \mathcal{X}_1$, and $\pi x = 0$ for all $x \in \mathcal{X}_2$.

of minimal degree such that $\xi^T \psi(\bar{A}) = 0$. Clearly, such a polynomial exists and has a degree less than or equal to $\dim(\mathcal{V}^*)$. For some polynomial ϕ with degree one less than that of ψ , we have $\psi(s) = \phi(s)(s - \lambda)$ where $\lambda \in \mathbb{C}$. Define $\eta = \phi^*(\bar{A})\xi$. By the definition of ψ , one gets $\eta \neq 0$ and

$$\eta^* \pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*} = \lambda \eta^*. \quad (19)$$

It follows from (18) that

$$\eta^* \pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{T}^*} = 0, \quad (20)$$

$$\eta^* \pi_{\mathcal{V}^*}(B + N^\alpha D) = 0. \quad (21)$$

By using $\mathcal{V}^* \subseteq \ker(C - DK)$ and (21), we get

$$\begin{aligned} \eta^* \pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*} \\ = \eta^* \pi_{\mathcal{V}^*}(A - BK + N^\alpha(C - DK)) \pi_{\mathcal{V}^*} \end{aligned} \quad (22)$$

$$= \eta^* \pi_{\mathcal{V}^*}(A + N^\alpha C - (B + N^\alpha D)K) \pi_{\mathcal{V}^*} \quad (23)$$

$$= \eta^* \pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{V}^*}. \quad (24)$$

It follows from (19), (20), and (24) that

$$\eta^* \pi_{\mathcal{V}^*}(A + N^\alpha C) = \lambda \eta^* \pi_{\mathcal{V}^*}. \quad (25)$$

Together with (21) and statement 1, this means that $\eta^* \pi_{\mathcal{V}^*} = 0$, i.e. $\eta \in (\mathcal{V}^*)^\perp$. It follows from the definition of η that $\xi \in (\mathcal{V}^*)^\perp$. Hence, $\mathcal{R}^\perp \subseteq (\mathcal{V}^*)^\perp$. Clearly, $\mathcal{R} \supseteq \mathcal{V}^*$.

2 \Rightarrow 3: Note that $\mathcal{T}^* \subseteq \langle A + N^\alpha C \mid \text{im}(B + N^\alpha D) \rangle$ for all $\alpha \in A$ due to (A.4) and (A.5). Since $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$ due to invertibility, it is enough to show that \mathcal{V}^* is contained in the sum of the subspaces $\langle (A + N^\alpha C) \mid \text{im}(B + N^\alpha D) \rangle$ over $\alpha \in A$. Let \mathcal{R}_α denote the subspace $\langle \pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*} \mid \text{im}(\pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{T}^*}) + \text{im}(\pi_{\mathcal{V}^*}(B + N^\alpha D)) \rangle$. Also let \mathcal{Q}_α denote the subspace $\langle (A + N^\alpha C) \mid \text{im}(B + N^\alpha D) \rangle$. Since $\mathcal{T}^* \subseteq \mathcal{Q}_\alpha \subseteq \mathcal{V}^* \oplus \mathcal{T}^*$, one gets $\mathcal{Q}_\alpha = \mathcal{T}^* \oplus (\mathcal{Q}_\alpha \cap \mathcal{V}^*)$. Hence,

$$\pi_{\mathcal{V}^*} \mathcal{Q}_\alpha = \mathcal{Q}_\alpha \cap \mathcal{V}^*. \quad (26)$$

Note that

$$\begin{aligned} \text{im } \pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{T}^*} \\ = \pi_{\mathcal{V}^*}(A + N^\alpha C) \mathcal{T}^* \\ \subseteq \pi_{\mathcal{V}^*}(A + N^\alpha C) \mathcal{Q}_\alpha \subseteq \pi_{\mathcal{V}^*} \mathcal{Q}_\alpha \stackrel{(26)}{=} \mathcal{Q}_\alpha \cap \mathcal{V}^*, \end{aligned} \quad (27)$$

$$\text{im } \pi_{\mathcal{V}^*}(B + N^\alpha D) \subseteq \pi_{\mathcal{V}^*} \mathcal{Q}_\alpha \stackrel{(26)}{=} \mathcal{Q}_\alpha \cap \mathcal{V}^*. \quad (28)$$

Also note that $\pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*} = \pi_{\mathcal{V}^*}(A + N^\alpha C - (B + N^\alpha D)K) \pi_{\mathcal{V}^*}$. Since both \mathcal{Q}_α and \mathcal{V}^* are $\pi_{\mathcal{V}^*}(A + N^\alpha C - (B + N^\alpha D)K) \pi_{\mathcal{V}^*}$ -invariant, $\mathcal{Q}_\alpha \cap \mathcal{V}^*$ is a $\pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*}$ -invariant subspace that contains both $\text{im } \pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{T}^*}$ and $\pi_{\mathcal{V}^*}(B + N^\alpha D)$. By definition, \mathcal{R}_α is the smallest of such spaces, hence $\mathcal{R}_\alpha \subseteq \mathcal{Q}_\alpha$. Note that $\sum_{\alpha \in A} \mathcal{R}_\alpha = \langle \pi_{\mathcal{V}^*}(A - BK) \pi_{\mathcal{V}^*} \mid \sum_{\alpha \in A} [\text{im}(\pi_{\mathcal{V}^*}(A + N^\alpha C) \pi_{\mathcal{T}^*}) + \text{im}(\pi_{\mathcal{V}^*}(B + N^\alpha D))] \rangle$. Therefore, we get $\mathcal{V}^* = \sum_{\alpha \in A} \mathcal{R}_\alpha \subseteq \sum_{\alpha \in A} \mathcal{Q}_\alpha = \sum_{\alpha \in A} \langle A + N^\alpha C \mid \text{im}(B + N^\alpha D) \rangle$.

3 \Rightarrow 1: Suppose that 3 holds. Let z be such that

$$z^*(A + N^\alpha C) = \lambda z^*, \quad z^*(B + N^\alpha D) = 0$$

for all $\alpha \in A$ and for some $\lambda \in \mathbb{C}$.

Then, $z \in (\langle A + N^\alpha C \mid \text{im}(B + N^\alpha D) \rangle)^\perp$ for all α . This means that $z \in (\sum_{\alpha \in A} \langle A + N^\alpha C \mid \text{im}(B + N^\alpha D) \rangle)^\perp$. Consequently, $z = 0$. \square

The second auxiliary lemma bridges Theorem 3.1 and Corollary 3.3.

Lemma 3.5. Suppose that $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. Let $A = \{1, 2, \dots, m\}$ and $M^\alpha = -E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0]\Pi^\alpha$ for each $\alpha \subseteq A$. Then, the following statements hold.

(1) $z^*A = \lambda z^*$, $z^*B = 0$, and $z^*E = 0$ if and only if $z^*(A + M^\alpha C) = \lambda z^*$, $z^*(B + M^\alpha D) = 0$ for all $\alpha \subseteq A$.

(2) The inequality system

$$\eta \geq 0, \quad (29a)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0, \quad (29b)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} E \\ F \end{bmatrix} \leq 0 \quad (29c)$$

has a solution (ξ, η) for a real number λ if and only if the inequality system

$$\begin{aligned} [z^T \ (w^\alpha)^T] \begin{bmatrix} A + M^\alpha C - \lambda I & B + M^\alpha D \\ C & D \end{bmatrix} = 0, \\ w^\alpha \in \mathcal{Y}_\alpha^* \end{aligned} \quad (30)$$

has a solution z and $\{w^\alpha\}$ for $\alpha \subseteq A$ and for a real number λ .

Proof. (1) Since $M^\alpha = -E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0]\Pi^\alpha$, the ‘only if’ part is evident. For the ‘if’ part, let z and λ be such that

$$z^*(A + M^\alpha C) = \lambda z^*, \quad (31a)$$

$$z^*(B + M^\alpha D) = 0, \quad (31b)$$

for all $\alpha \subseteq A$. Take $\alpha = \emptyset$. This gives,

$$z^*A = \lambda z^*, \quad (32a)$$

$$z^*B = 0. \quad (32b)$$

Then, (31) and (32) yield

$$0 = z^*M^\alpha C = -z^*E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0]\Pi^\alpha C, \quad (33a)$$

$$0 = z^*M^\alpha D = -z^*E_{\bullet\alpha}[F_{\alpha\alpha}^{-1} \ 0]\Pi^\alpha D \quad (33b)$$

for all index sets α . It follows from invertibility of $D + C(sI - A)^{-1}B$ that $[C \ D]$ is of full row rank. Further, Π^α and $F_{\alpha\alpha}$ are both invertible. Therefore, (33) implies that

$$z^*E = 0. \quad (34)$$

(2) Note first that

$$[z^T \ (w^\alpha)^T] \begin{bmatrix} A + M^\alpha C - \lambda I & B + M^\alpha D \\ C & D \end{bmatrix} = 0$$

for all $\alpha \subseteq \{1, 2, \dots, m\}$

if and only if

$$[z^T \ (w^\theta)^T] \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0, \quad (w^\alpha)^T = (w^\theta)^T - z^T M^\alpha$$

for all $\alpha \subseteq \{1, 2, \dots, m\}$.

This is due to the fact that $[C \ D]$ is of full row rank. For the rest, it is enough to show that $(w^\theta)^T - z^T M^\alpha \in \mathcal{Y}_\alpha^*$ for all $\alpha \subseteq \{1, 2, \dots, m\}$ if and only if

$$w^\theta \geq 0 \quad \text{and} \quad [z^T \ (w^\theta)^T] \begin{bmatrix} E \\ F \end{bmatrix} \leq 0. \quad (35)$$

Since $\mathcal{Y}_\alpha^* = \{y \in \mathbb{R}^m \mid y^T = v^T T^\alpha \Pi^\alpha, v \geq 0\}$, the former is equivalent to

$$[z^T \ (w^\theta)^T] \begin{bmatrix} -M^\alpha \\ I \end{bmatrix} = v^T T^\alpha \Pi^\alpha \quad (36)$$

for some $v \geq 0$. Note that

$$\begin{aligned} -M^\alpha (\Pi^\alpha)^{-1} (T^\alpha)^{-1} &= [E_{\bullet\alpha} F_{\alpha\alpha}^{-1} \ 0] \Pi^\alpha (\Pi^\alpha)^{-1} (T^\alpha)^{-1} \\ &= [E_{\bullet\alpha} F_{\alpha\alpha}^{-1} \ 0] \begin{bmatrix} -F_{\alpha\alpha} & 0 \\ -F_{\alpha^c\alpha} & I \end{bmatrix} \\ &= [-E_{\bullet\alpha} \ 0]. \end{aligned}$$

Also note that

$$\Pi^\alpha = \begin{bmatrix} I_{\alpha\bullet} \\ I_{\alpha^c\bullet} \end{bmatrix}$$

and thus

$$(\Pi^\alpha)^{-1} = [I_{\bullet\alpha} \ I_{\bullet\alpha^c}].$$

This results in

$$(\Pi^\alpha)^{-1} (T^\alpha)^{-1} = [I_{\bullet\alpha} \ I_{\bullet\alpha^c}] \begin{bmatrix} -F_{\alpha\alpha} & 0 \\ -F_{\alpha^c\alpha} & I \end{bmatrix} = [-F_{\bullet\alpha} \ I_{\bullet\alpha^c}].$$

Therefore, we get

$$z^T E_{\bullet\alpha} + (w^\theta)^T F_{\bullet\alpha} \leq 0 \quad \text{and} \quad (w^\theta)^T \geq 0$$

for all $\alpha \subseteq \{1, 2, \dots, m\}$

by right-multiplying (36) by the inverse of $T^\alpha \Pi^\alpha$. This, in turn, is equivalent to (35). \square

3.1. Proof of Theorem 3.1

Lemma 3.5(1), together with Lemma 3.4, implies that the first conditions of Theorem 3.1 and Corollary 3.3 are equivalent. The equivalence of the second conditions of these two follow from Lemma 3.5(2).

3.2. Particular cases

The two particular cases that are mentioned earlier can be recovered from Theorem 3.1 as follows.

3.2.1. Linear systems

If we take $C = 0$, $D = I$, $E = 0$, and $F = I$ as in (8), the two conditions of Theorem 3.1 boil down to

- (1) the pair (A, B) is controllable, and
- (2) the system of inequalities

$$\eta \geq 0, \quad (37a)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} A - \lambda I & B \\ 0 & I \end{bmatrix} = 0, \quad (37b)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} 0 \\ I \end{bmatrix} \leq 0 \quad (37c)$$

admits no nonzero solution (ξ, η) for a real number λ .

Note that (37a) and (37c) imply that $\eta = 0$. This means that if (A, B) is controllable then (37b) is satisfied only if $\xi = 0$. Hence, we recover the case of linear systems.

3.2.2. Linear systems with nonnegative inputs:

If we take $C = 0$, $D = I$, $E = B$, $F = I$ as in (10), the two conditions of Theorem 3.1 boil down to

- (1) the pair (A, B) is controllable, and
- (2) the system of inequalities

$$\eta \geq 0, \quad (38a)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} A - \lambda I & B \\ 0 & I \end{bmatrix} = 0, \quad (38b)$$

$$[\xi^T \ \eta^T] \begin{bmatrix} B \\ I \end{bmatrix} \leq 0 \quad (38c)$$

admits no nonzero solution (ξ, η) for a real number λ .

Note that (38c) is satisfied as equality due to (38b). Let (ξ, η) be a nonzero solution of (38) for some real number λ . Then, the condition (11) is violated for $z = -\xi$ and the same λ . Conversely, if z violates (11) for some real number λ then $(\xi, \eta) = (-z, B^T z)$ is a nonzero solution of (38) for the same λ . Hence, we establish the equivalence of the second condition above and the second condition that is presented in (11).

4. Conclusions

This paper studied controllability problem for the linear complementarity class of hybrid systems. These systems are closely related to the so-called CLSs. By exploiting this connection, together with the special structure of complementarity systems, we derived algebraic necessary and sufficient conditions for controllability. We also showed that Kalman's and Bramer's

controllability results for linear systems can be recovered from our main theorem. Our treatment employed a mixture of methods from both mathematical programming and geometric control theory. Obvious question is how one can utilize these techniques in order to establish necessary and/or sufficient conditions for the (feedback) stabilizability problem.

Appendix A. Geometric control theory

Consider the linear system $\Sigma(A, B, C, D)$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (\text{A.1a})$$

$$y(t) = Cx(t) + Du(t), \quad (\text{A.1b})$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and the matrices A, B, C, D are of appropriate sizes.

We define the *controllable subspace* as $\langle A \mid \text{im} B \rangle := \text{im} B + A \text{im} B + \cdots + A^{n-1} \text{im} B$. Note that

$$\langle A \mid \text{im} B \rangle = \langle A - BK \mid \text{im} B \rangle \quad (\text{A.2})$$

for all matrices K with the appropriate sizes.

We say that a subspace \mathcal{V} is *output-nulling controlled invariant* if for some matrix K the inclusions $(A - BK)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C - DK)$ hold. As the set of such subspaces is nonempty and closed under subspace addition, it has a maximal element $\mathcal{V}^*(\Sigma)$. Whenever the system Σ is clear from the context, we simply write \mathcal{V}^* . The notation $\mathcal{H}(\mathcal{V})$ stands for the set $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK)\}$.

Dually, we say that a subspace \mathcal{T} is *input-containing conditioned invariant* if for some matrix L the inclusions $(A - LC)\mathcal{T} \subseteq \mathcal{T}$ and $\text{im}(B - LD) \subseteq \mathcal{T}$ hold. As the set of such subspaces is nonempty and closed under the subspace intersection, it has a minimal element $\mathcal{T}^*(\Sigma)$. Whenever the system Σ is clear from the context, we simply write \mathcal{T}^* . The notation $\mathcal{L}(\mathcal{T})$ stands for the set $\{L \mid (A - LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \text{im}(B - LD) \subseteq \mathcal{T}\}$.

We sometimes write $\mathcal{V}^*(A, B, C, D)$ or $\mathcal{T}^*(A, B, C, D)$ to make the dependence on (A, B, C, D) explicit.

The following properties are among the standard facts of geometric control theory:

$$\mathcal{V}^*(A - BK, B, C - DK, D) = \mathcal{V}^*(A, B, C, D) \quad (\text{A.3})$$

for all $K \in \mathcal{H}$,

$$\mathcal{T}^*(A - LC, B - LD, C, D) = \mathcal{T}^*(A, B, C, D) \quad (\text{A.4})$$

for all $L \in \mathcal{L}$,

$$\mathcal{T}^* \subseteq \langle A \mid \text{im} B \rangle. \quad (\text{A.5})$$

It is well-known (see e.g. [1]) that the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if, and only if,

$$(1) \quad \mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n,$$

(2) $\text{col}(B, D)$ is of full column rank, and

(3) $[C \ D]$ is of full row rank.

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